Quiz 1

Name: Solutions

Math 240 - Calculus III
Thursday, 05/26/2011

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (2 points) Is the line $y = x + 1$ a subspace of $\mathbb{R}^2$?

   Consider 2 points on the line $(0,1)$ and $(1,2)$. Their sum $(1,3)$ is not on the line, so the line does not satisfy the additive closure property, and as such it is NOT a subspace of $\mathbb{R}^2$.

2. (2 points) Are the vectors $v_1 = \langle 1,0,0,0\rangle$, $v_2 = \langle 0,1,0,0\rangle$, $v_3 = \langle 0,0,1,1\rangle$, and $v_4 = \langle 3,5,2,2\rangle$ linearly independent or dependent?

   Since $3v_1 + 5v_2 + 2v_3 - v_4 = 0$, they are Linearly Dependent.

3. (2 points) Let A and B be n-by-n symmetric matrices. Must the product AB be symmetric as well?

   NO! Consider $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

   If, when first looking at this problem, you have no idea, then recall that AB being symmetric means that $(AB)^T = AB$. But $(AB)^T = B^T A^T = BA$, which we said in general is not equal to AB. This doesn't actually prove that AB is not symmetric, since we don't know when $AB = BA$, but it should give you the intuition to look for a counterexample.

4. (a) (1 point) Write the augmented matrix for the following system of linear equations:

$$\begin{aligned} x+y+z&=3\\ x-y-z&=-1\\ 3x+y+z&=5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & 5 \end{array}\right)$$

(b) (1 point) Put the augmented matrix in row-echelon form (you do not have to make the leading non-zero terms 1).

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & -1 & -1 \\ 3 & 1 & 1 & 5 \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -2 & -4 \\ 0 & -2 & -2 & -4 \end{array}\right)$$

$$\xrightarrow{r_3-r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(c) (2 points) Solve the system of linear equations.

$-2y - 2z = -4$, so $y + z = 2$. Let $y = t$, $z = 2 - t$

$x + y + z = 3$, so $x = 1$.

Solution set : $\{ (1, t, 2-t) \mid t \in \mathbb{R} \}$

Quiz 2
Math 240 - Calculus III
Thursday, 06/02/2011

Name: Solutions

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (2 points) Suppose the system $AX = B$ in n unknowns is inconsistent. Determine if each of the following is always true, sometimes true, or never true. Briefly explain.

   (a) Rank $(A) < n$. Always True

   If $AX = B$ is inconsistent, then $\text{rank}(A) < \text{rank}(A|B)$.
   $\text{rank}(A|B) \leq n$, so necessarily $\text{rank}(A) < n$.

   (b) Rank $(A \mid B) < n$. Sometimes True

   The key is that $\text{rank}(A) < \text{rank}(A|B)$. There are no other restrictions on $\text{rank}(A|B)$.

2. (4 points) Let

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 4 & 0 & 1 \\ 8 & 6 & 1 \end{pmatrix}$$

Compute $\det(A^{-1})$, or show that A^{-1} does not exist.

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(A) = -2(4-8) + 24 = 32$$

$$\text{So } \det(A^{-1}) = \frac{1}{32}$$

3. (4 points) Let

$$A = \begin{pmatrix} -4 & -3 \\ 8 & 6 \end{pmatrix}$$

Find invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, or show that no such matrices exist.

First we find eigenvalues of A:

$$\det(A - \lambda I) = (-4-\lambda)(6-\lambda) + 24$$
$$= -24 - 6\lambda + 4\lambda + \lambda^2 + 24$$
$$= \lambda^2 - 2\lambda = \lambda(\lambda - 2) \qquad \lambda = 0, 2$$

$\underline{\lambda = 0}$

$$\begin{pmatrix} -4 & -3 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -4x_1 - 3x_2 = 0 \\ x_1 = 3 \quad x_2 = -4 \end{matrix} \qquad v = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} -6 & -3 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -6x_1 - 3x_2 = 0 \\ x_1 = 1 \quad x_2 = -2 \end{matrix} \qquad v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

since we have 2 linearly independent eigenvectors, A is diagonalizable (in fact, just knowing that we have 2 distinct eigenvalues already tells us that A is diagonalizable.)

$$P = \begin{pmatrix} 3 & 1 \\ -4 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Quiz 3

Name: Solutions

Math 240 - Calculus III
Thursday, 06/16/2011

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (2 points) Identify the singular point(s) of the following differential equation, and classify each as regular or irregular:

$$(x^2-9)^2y'' + (x+3)y' + 2y = 0.$$

put in regular form: $y'' + \underbrace{\frac{(x+3)}{(x+3)^2(x-3)^2}}_{P(x)} y' + \underbrace{\frac{2}{(x+3)^2(x-3)^2}}_{Q(x)} y = 0$

singular points are ± 3.

since $(x+3)P(x)$ and $(x+3)^2Q(x)$ at both analytic at $x=-3$,
-3 is a regular singular point.

since $(x-3)P(x)$ is not analytic at $x=3$, 3 is a irregular singular point.

2. (8 points) Find two values of r for which there exist solutions of the form $\sum c_n x^{n+r}$ to the differential equation

$$2xy'' - y' + 2y = 0.$$

Furthermore, for each r, find the recurrence relation for the coefficients of the corresponding solution.

let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$

$$2xy'' - y' + 2y = \sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} 2c_n x^{n+r}$$

$$= x^r\left(\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n-1} - \sum_{n=0}^{\infty} c_n(n+r)x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n\right)$$

$$= x^r\left(2c_0r(r-1)x^{-1} - c_0rx^{-1} + \sum_{n=1}^{\infty} \underbrace{\left[2c_n(n+r)(n+r-1) - c_n(n+r)\right]x^{n-1}}_{k=n-1,\ n=k+1} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{k=n}\right)$$

$$= x^r\left(c_0r(2r-2-1)x^{-1} + \sum_{k=0}^{\infty}\left(2c_{k+1}(k+1+r)(k+r) - c_{k+1}(k+1+r) + 2c_k\right)x^k\right)$$

$$= x^r\left(c_0r(2r-3)x^{-1} + \sum_{k=0}^{\infty}\left(c_{k+1}(k+1+r)(2k+2r-1) + 2c_k\right)x^k\right)$$

indicial equation: $r(2r-3)=0 \qquad r=0, \frac{3}{2}$

recurrence relation: $c_{k+1} = \dfrac{-2c_k}{(k+1+r)(2k+2r-1)}$

$r=0: \quad c_{k+1} = \dfrac{-2c_k}{(k+1)(2k-1)}$ $\qquad r=\frac{3}{2}: \quad c_{k+1} = \dfrac{-2c_k}{(k+\frac{5}{2})(2k+2)}$

$$= \frac{-2c_k}{(2k+5)(k+1)}$$

Quiz 4

Name: Solutions

Math 240 - Calculus III
Thursday, 06/23/2011

Note: *In order to receive full credit, you must show work that justifies your answer.*

1. (2 points) Determine if each of the following expressions is meaningful. If so, state whether it is a scalar function or vector field. If not, explain why not.

   (a) curl(curl F), where F is a vector field

   meaningful, vector field

   (b) $\underbrace{(\text{curl}(\text{grad } f))}_{\text{vector}} \times \underbrace{(\text{div}(\text{grad } f))}_{\text{scalar}}$, where f is a scalar function

   not meaningful, because the cross product is only defined between vectors.

2. (3 points) Find all points (if any) on the surface $\underbrace{x^2+4x+y^2+z^2-2z}_{F(x,y,z)} = 11$ at which the tangent plane is parallel to the xy-plane.

the tangent plane is parallel to the xy-plane when their normal vectors are parallel, i.e. when $\nabla F = c\langle 0,0,1\rangle$ for some $c \neq 0$.

$\nabla F = \langle 2x+4,\ 2y,\ 2z-2\rangle = \langle 0, 0, c\rangle$

So $2x+4=0 \Rightarrow x=-2$

$2y=0 \Rightarrow y=0$

$2z-2=c \Rightarrow z=\frac{c+2}{2}$

Substitute back onto $x^2+4x+y^2+z^2-2z=11$:

$$(-2)^2+4(-2)+\left(\frac{c+2}{2}\right)^2-c-2=11$$

$$-4+\frac{c^2}{4}+c+1-c-2=11$$

$$c^2=64$$

$$c=\pm 8$$

$$\Rightarrow z=5,-3$$

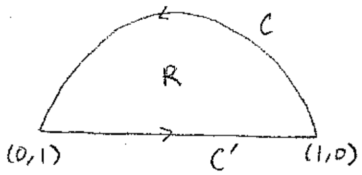
So the desired points are $(-2,0,5)$ and $(-2,0,-3)$

3. (5 points) Compute the line integral

$$\int_C \underbrace{\left(\frac{\sin(2y)-2y}{4}+x^2\right)}_{P} dx + \underbrace{(x\cos^2 y)}_{Q}dy,$$

where C is the top half of the circle $x^2+y^2=1$, traversed counterclockwise.

Possibly useful (or useless) identities: $\sin(2\theta) = 2(\sin\theta)(\cos\theta)$, $\cos(2\theta) = 2\cos^2\theta - 1$



by Green's theorem,

$$\oint_{C \cup C'} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

so $\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA - \int_{C'} P dx + Q dy$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \left[\cos^2 y - \frac{1}{4}(2\cos(2y) - 2)\right] dA$$

$$= \iint_R \left[\cos^2 y - \frac{1}{4}(4\cos^2 y - 4)\right] dA$$

$$= \iint_R (\cos^2 y - \cos^2 y + 1) dA = \text{Area}(R) = \frac{\pi}{2}$$

parametrize C' as $\langle x, y\rangle$, $-1 \le x \le 1$, $dy = 0$

$$\int_{C'} P dx + Q dy = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right] = \frac{2}{3}$$

thus $\int_C \left(\frac{\sin(2y) - 2y}{4} + x^2\right) dx + (x\cos^2 y) dy = \frac{\pi}{2} - \frac{2}{3}$

Name: Solutions

Math 240 Midterm
June 13 2011, 1:00-3:00pm
Instructor: Shanshan Ding

No calculators, books, or notes may be used except for a "cheat sheet" made out of one side of an 8.5-by-11 paper.

There are 12 questions, the point value of each is indicated below. There is also a bonus question worth up to 5 points.

For each question, write your final answer in the designated box on each page. You must also show all your work and justify your answers; answers with no work or incorrect work will not receive full credit.

For grading purposes only (do not fill in):

1) ________ / 8

2) ________ / 8

3) ________ / 8

4) ________ / 8

5) ________ / 8

6) ________ / 8

Bonus ________

7) ________ / 8

8) ________ / 8

9) ________ / 8

10) ________ / 8

11) ________ / 4

12) ________ / 16

Total ________ **/ 100**

1. Solve the following system of equations, or show that no solution exists:

$$\begin{aligned} x+y+z &= 2 \\ 3x+y+z &= 4 \\ 4x-4y-4z &= 0 \end{aligned}$$

use your favorite method

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 3 & 1 & 1 & 4 \\ 4 & -4 & -4 & 0 \end{array}\right) \xrightarrow{\frac{1}{4} r_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 1 & 1 & 4 \\ 1 & -1 & -1 & 0 \end{array}\right) \xrightarrow[r_3-r_1]{r_2-3r_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{array}\right)$$

at this point we have

$$\left.\begin{aligned} x+y+z &= 2 \\ -2y-2z &= -2 \end{aligned}\right\} \quad \begin{aligned} y+z &= 1 \\ z &= t \\ y &= 1-t \end{aligned} \qquad \begin{aligned} x+(1-t)+t &= 2 \\ x &= 1 \end{aligned}$$

Put your answer here:

$x=1, \ y=1-t, \ z=t \quad (\text{or } y=t, \ z=1-t)$

2. Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2a_1 & 2a_2 & 2a_3 & a_3 + 2a_4 \\ 2c_1 & 2c_2 & 2c_3 & c_3 + 2c_4 \\ 2b_1 & 2b_2 & 2b_3 & b_3 + 2b_4 \\ 2d_1 & 2d_2 & 2d_3 & d_3 + 2d_4 \end{pmatrix}.$$

If $\det(A) = 1$, find $\det(AB)$.

$$\det(B) = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 & a_3 + 2a_4 \\ 2c_1 & 2c_2 & 2c_3 & c_3 + 2c_4 \\ 2b_1 & 2b_2 & 2b_3 & b_3 + 2b_4 \\ 2d_1 & 2d_2 & 2d_3 & d_3 + 2d_4 \end{vmatrix} = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 & 2a_4 \\ 2c_1 & 2c_2 & 2c_3 & 2c_4 \\ 2b_1 & 2b_2 & 2b_3 & 2b_4 \\ 2d_1 & 2d_2 & 2d_3 & 2d_4 \end{vmatrix}$$

$$= 2^4 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= -16 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= -16 \det(A) = -16$$

$$\det(AB) = \det(A)\det(B) = (1)(-16) = -16$$

$\det(AB) = -16$

3. For what value(s) of k are the vectors $\langle 1, 0, -2\rangle$, $\langle 0, 2, 3\rangle$, and $\langle 3, 2, k\rangle$ linearly dependent?

put these as the rows of a matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 3 & 2 & k \end{pmatrix}$$

the rows are linearly independent iff rank $(A) = 3$, which happens iff $\det(A) \neq 0$.

Since we want to know when the rows are linearly dependent, we set $\det(A) = 0$

$\det(A) = (2k-6)-2(-6) = 2k+6$

$k = -3$

$k = -3$

4. Find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

or show that A^{-1} does not exist.

Elimination method works well here (det (A) is obviously nonzero, so A^{-1} exists):

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_2 \,\&\, r_3]{\text{switch}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{r_2 - r_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

Put your answer here:

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

5. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

$\det(A-\lambda) = (-\lambda)^2 + 4 = \lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$\underline{\lambda = 2i}$

$\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2ix_1 + 2x_2 = 0$

$-2x_1 - 2ix_2 = 0$

bottom eq. is just the top eq. multiplied by i

let $x_1 = 1$

$2x_2 = 2i \quad x_2 = i$

$K = \begin{pmatrix} 1 \\ i \end{pmatrix}$

the eigenvector for $\bar{\lambda} = -2i$ is just $\bar{K} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Put your answer here (if there are multiple eigenvalues/eigenvectors, be clear on which corresponds to which):

$\lambda_1 = 2i, \ K_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\qquad \lambda_2 = -2i, \ K_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

6. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, or show that no such matrices exist.

$\det(A - \lambda I) = (1-\lambda)[(-\lambda)^2 - 1] = (1-\lambda)(\lambda-1)(\lambda+1)$

$\lambda = -1, 1$ (with multiplicity 2)

$\underline{\lambda = -1}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad k = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad k = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

since A has 3 linearly independent eigenvectors, A is diagonalizable.

Put your answer here:

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

D is unique up to order of the eigenvalues, P is not unique, due to choice of eigenvectors

7. Find the general solution to the differential equation

$$y^{(7)} - 6y^{(6)} + 14y^{(5)} - 20y^{(4)} + 25y''' - 22y'' + 12y' - 8y = -16x.$$

[Hint: $m^7 - 6m^6 + 14m^5 - 20m^4 + 25m^3 - 22m^2 + 12m - 8 = (m-2)^3(m^2+1)^2$.]

from the hint, the general solution of the associated homogeneous equation is

$y_c = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 \cos x + c_5 \sin x + c_6 x \cos x + c_7 x \sin x$

the particular solution is of the form

$$y_p = Ax + B$$

then $y_p' = A$ are all higher derivatives are 0

solve for A and B: $12A - 8(Ax+B) = -16x$

$$\left.\begin{array}{r} -8x = -16x \\ 12A - 8B = 0 \end{array}\right\} \begin{array}{l} A = 2 \\ B = 3 \end{array}$$

so $y_p = 2x+3$

and the general solution is $y(x) = y_c + y_p$

$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4 \cos x + c_5 \sin x + c_6 x \cos x + c_7 x \sin x + 2x + 3$

8. Let $y(x)$ be the solution to the initial value problem

$$x^2y'' + xy' + y = 0, y(1) = 5, y'(1) = 3.$$

What is $y(e^\pi)$?

This is a Cauchy-Euler equation:

auxiliary equation is $m^2 + (1-1)m + 1 = m^2+1 = 0 \quad m = \pm i$

general solution is $y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

Solving the IVP: $y(1) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 5$

$$y'(x) = \frac{-c_1 \sin(\ln x)}{x} + \frac{c_2 \cos(\ln x)}{x}$$

$$y'(1) = -c_1 \sin(0) + c_2 \cos(0) = c_2 = 3$$

Hence $y(x) = 5\cos(\ln x) + 3\sin(\ln x)$

and $y(e^\pi) = 5\cos(\pi) + 3\sin(\pi) = -5$

p.s. this is almost exactly one of the homework problems!

$y(e^\pi) = -5$

9. Find the general solution to the following system of differential equations:

$$X' = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{A} X$$

$$\det(A - \lambda I) = (1-\lambda)\left[(3-\lambda)(1-\lambda)+1\right]$$

$$= (1-\lambda)(\lambda^2 - 4\lambda + 4)$$

$$= (1-\lambda)(\lambda-2)^2 \qquad \lambda = 1, 2 \text{ (with multiplicity 2)}$$

$\underline{\lambda = 1}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad K_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

only one eigenvector corresponding to $\lambda = 2$, so need to solve

$$(A - \lambda I)P = K_2 : \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

the general solution is then

$$X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + c_3\left[K_2 t e^{\lambda_2 t} + P e^{\lambda_2 t}\right]$$

$$X(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$$

again, the choice of the vectors are not unique

10. Suppose that e^x and $\sin(2x)$ are solutions to a third order linear differential equation that is homogeneous and has constant coefficients. Find the differential equation.

since $\sin(2x)$ is a solution, $\cos(2x)$ is also a solution. this means that the roots of the auxiliary equation, which is of the form

$$am^3 + bm^2 + cm + d = 0,$$

(for a 3rd order ODE with constant coefficients), are exactly 1 and ± 2i

thus the auxiliary polynomial is

$$(m-1)(m^2+4) = m^3 - m^2 + 4m - 4.$$

since this is a homogeneous ODE, it must be

$$y''' - y'' + 4y' - 4y = 0$$

Put your answer here:

$$y''' - y'' + 4y' - 4y = 0$$

11. A force of 12 N stretches a spring 2 m. An object whose mass is 2 kg is initially released from 1 m below the equilibrium position with an upward velocity of 3 m/s, and the subsequent motion takes place in a medium that offers a damping force numerically equal to 4 times the instantaneous velocity. Additionally, an external force of $f(t) = 8e^t$ is applied to the system. Write down the initial value problem whose solution is the equation of motion for this object on the spring. **Do not solve unless you are bored.**

[Note: 1 N = (1 kg · m)/s^2.]

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$

$m = 2$ $\qquad x(0) = 1$

$\beta = 4$ $\qquad x'(0) = -3$

$k = \frac{12}{2} = 6$

$f(t) = 8e^t$

Put your answer here:

$$2\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 8e^t$$

or

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = 4e^t$$

with initial conditions $x(0) = 1$

$x'(0) = -3$

12. For each of the following four statements, given that the first half of the statement is satisfied, determine if the second half is always, sometimes, or never true (circle one). Briefly justify your answers; an answer with no explanation will receive no credit.

(a) A homogeneous system has 12 equations and 16 unknowns. The solution to this system contains 4 paramaters.

ALWAYS [✓ SOMETIMES] NEVER

it's the rank that matters, not the number of equations.

in the system $AX = B$, where A is 12×16, $\text{rank}(A) \leq 12$.

if $\text{rank}(A) = 12$, the solution contains 4 parameters

if $\text{rank}(A) < 12$, the solution contains > 4 parameters.

(b) A is an n-by-n matrix such that $A^2 = A$. A is the n-by-n identity matrix.

ALWAYS [✓ SOMETIMES] NEVER

A could also be the zero matrix.

Better question: if $A^2 = A$, must A be either the identity or the zero matrix?

(c) A is a 3-by-3 matrix such that A^3 is the zero matrix. Zero is the only eigenvalue of A.

[✓ ALWAYS] SOMETIMES NEVER

let λ be an eigenvalue of A, and let K be the corresponding eigenvector.

then $A^3K = A^2\lambda K = \lambda A^2 K = \lambda A \lambda K = \lambda^2 A K = \lambda^3 K$, which is $\vec{0}$, since

A is the zero matrix. since $K \neq \vec{0}$ by definition, $\lambda^3 = 0 \Rightarrow \lambda = 0$

(d) A differential equation is of the form $y'' + ky' - y = \cos(3x)$, where k is a positive constant. The form of a particular solution to this differential equation is $Ax\cos(3x) + Bx\sin(3x)$.

ALWAYS SOMETIMES [✓ NEVER]

the auxiliary polynomial is $m^2 + km - 1$, which has roots $\frac{-k \pm \sqrt{k^2+4}}{2}$,

which are real. so sine and cosine does not appear in the solution of

$y'' + ky' - y = 0$, which means that the form of the particular solution

is just $A\cos(3x) + B\sin(3x)$.

Bonus: Suppose A is a 3-by-3 matrix such that

$$A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad A\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}5\\10\\0\end{pmatrix},\quad \text{and}\quad A\begin{pmatrix}-1\\-2\\4\end{pmatrix}=\begin{pmatrix}7\\14\\-28\end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A^{-1}, or show that A^{-1} does not exist.

[If you wish to use any theorems that we did not cover in class, you must prove these theorems first.]

The above conditions tell us that the eigenvalues of A are $1, 5, -7$, with ~~eigen~~eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\\4\end{pmatrix}$, respectively.

If $AK=\lambda K$, A is invertible (which we know it is here because 0 is not an eigenvalue), and $\lambda \neq 0$ (which must be the case anyway for A to be invertible), then $\lambda^{-1}A^{-1}AK = \lambda^{-1}A^{-1}\lambda K = \lambda^{-1}\lambda A^{-1}K$

$$\Rightarrow \quad \lambda^{-1}K = A^{-1}K$$

thus the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, with the same eigenvectors as A!

Put your answer here (if there are multiple eigenvalues/eigenvectors, be clear on which corresponds to which):

$\lambda_1 = 1$, $K_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $\lambda_2 = \frac{1}{5}$, $K_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}$; $\lambda_3 = -\frac{1}{7}$, $K_3 = \begin{pmatrix}-1\\-2\\4\end{pmatrix}$

Name: Solutions

Math 240 Final Exam
June 30 2011, 1:00-3:00pm
Instructor: Shanshan Ding

No calculators, books, or notes may be used except for a "cheat sheet" made out of both sides of an 8.5-by-11 paper.

There are 10 questions, each worth 10 points. In addition, there is an optional question that will be used for extra credit considerations toward your overall grade.

For each question, write your final answer in the designated box on each page. Unless otherwise indicated, you must also show all your work and justify your answers; answers with no work or incorrect work will not receive full credit.

Possibly useful (or useless) identities: $2\sin^2\theta = 1 - \cos(2\theta)$, $2\cos^2\theta = 1 + \cos(2\theta)$.

For grading purposes only (do not fill in):

1) ________ / 10
2) ________ / 10
3) ________ / 10
4) ________ / 10
5) ________ / 10
6) ________ / 10
7) ________ / 10
8) ________ / 10
9) ________ / 10
10) ________ / 10

Total ________ / 100

1. For what value(s) of a does the following system have no solution?

$$\begin{aligned} x+2y+2z &= 1 \\ 2x+ay+z &= 1 \\ y+az &= 1 \end{aligned}$$

let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & a & 1 \\ 0 & 1 & a \end{pmatrix}$ if the system has no solution, $\det A = 0$

$\det A = a^2 - 1 - 2(2a) + 2(2) = a^2 - 4a + 3 = (a-3)(a-1)$ $a = 3, 1$

but wait! $\det A = 0$ could also mean that there are infinitely many solutions!

so now we row-reduce $(A \mid B)$

if $a = 3$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & 1 & 3 & 1 \end{array}\right) \xrightarrow{r_3 + r_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ consistent infinitely many solutions

if $a = 1$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{r_3 + \frac{1}{3}r_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 4/3 \end{array}\right)$$ inconsistent no solution

$a =$ 1

2. For a square matrix A, let $(A^k)_{ij}$ denote the ij-th entry of the k-th power of A. We say that A^k converges to the matrix M if $\lim_{k\to\infty}(A^k)_{ij} = M_{ij}$ for each ij. What does A^k converge to, if A is a 2-by-2 matrix such that

$$\underbrace{A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\end{pmatrix}}_{\frac{1}{2}\text{ is an eigenvalue}} \quad\text{and}\quad \underbrace{A\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/4\end{pmatrix}}_{1/4\text{ is an eigenvalue}}?$$

thus $A = PDP^{-1}$, where $D = \begin{pmatrix}1/4 & 0\\ 0 & 1/2\end{pmatrix}$

then $A^k = PD^kP^{-1}$, where $D^k = \begin{pmatrix}(1/4)^k & 0\\ 0 & (1/2)^k\end{pmatrix}$

since $\lim_{k\to\infty} D^k = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, $\lim_{k\to\infty} A^k = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

A^k converges to $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

3. Find the general solution of

$$x^3y''' + xy' - y = 0.$$

Cauchy - Euler equation : assume $y = x^m$

$y' = mx^{m-1}$ $y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

substitute into the DE : $x^3 m(m-1)(m-2)x^{m-3} + x\,m x^{m-1} - x^m$

$$= x^m \left(\underbrace{m(m-1)(m-2) + m - 1} \right)$$

$$(m-1)\left(m(m-2)+1 \right) = 0$$

$$(m-1)(m^2 - 2m + 1) = (m-1)^3 = 0 \qquad m=1 \text{ (multiplicity 3)}$$

$y(x) = c_1 x + c_2 x \ln x + c_3 x (\ln x)^2$

4. Let $x(t)$ and $y(t)$ be solutions to the system of differential equations

$$x' = x + 2y$$
$$y' = 3x + 2y$$

for which

$$\lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = 0.$$

If $x(0) = 1$, what is $y(1)$?

$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ $\quad$ $\det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = 2 - 3\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1)$

$\lambda = 4, -1$

$\underline{\lambda = 4}$

$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u_1 = 2$, $u_2 = 3$

$\underline{\lambda = -1}$

$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u_1 = 1$, $u_2 = -1$

general solution is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}$, i.e. $x(t) = c_1 e^{-t} + 2c_2 e^{4t}$

$y(t) = -c_1 e^{-t} + 3c_2 e^{4t}$

since $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$, c_2 must be 0, so $x(t) = c_1 e^{-t}$ and $y(t) = -c_1 e^{-t}$

since $x(0) = 1$, $c_1 = 1$, and thus $y(t) = -e^{-t}$

$\boxed{y(1) = -e^{-1}}$

5. Find the series solution centered at zero, up to the x^5 term, of the differential equation

$$(x^2 - 1)y'' - 2y = 0$$

that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 1$.

$x = 0$ is an ordinary point, so series solutions are of the form $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

substitute into the DE : $(x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=0}^{\infty} c_n x^n = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=0}^{\infty} c_n x^n$$

$$= -2c_2 - 6c_3 x - 2c_0 - 2c_1 x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} - \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2,\ n=k+2} - \underbrace{\sum_{n=2}^{\infty} 2c_n x^n}_{k=n}$$

$$= (-2c_0 - 2c_2) + (-2c_1 - 6c_3)x + \underbrace{\sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=2}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=2}^{\infty} 2c_k x^k}_{\sum_{k=2}^{\infty}\left(k(k-1)c_k - (k+2)(k+1)c_{k+2} - 2c_k\right)x^k}$$

$$k(k-1) - 2 = k^2 - k - 2 = (k-2)(k+1)$$

$$= \sum_{k=2}^{\infty}\left((k-2)(k+1)c_k - (k+2)(k+1)c_{k+2}\right)x^k$$

recurrence relation : $c_{k+2} = \dfrac{(k-2)\cancel{(k+1)}c_k}{(k+2)\cancel{(k+1)}}$

$-2c_0 - 2c_2 = 0 \Rightarrow c_2 = -c_0$

$-2c_1 - 6c_3 = 0 \Rightarrow c_3 = -\frac{c_1}{3}$

$y(0) = 2 \Rightarrow c_0 = 2$

$y'(0) = 1 \Rightarrow c_1 = 1$

$c_2 = -2 \qquad c_3 = -\frac{1}{3}$

$$c_4 = \frac{(2-2)c_2}{(2+2)} = 0$$

$$c_5 = \frac{(3-2)c_3}{(3+2)} = -\frac{1}{15}$$

$y(x) = 2 + x - 2x^2 - \frac{1}{3}x^3 - \frac{1}{15}x^5 \ldots$

6. Let C be the curve $r(t) = \langle \cos^3 t, \sin^3 t \rangle$ as t increases from 0 to $\pi/2$, and let $F = \langle \underbrace{y \cos xy - 1}_{P}, \underbrace{1 + x \cos xy}_{Q} \rangle$. Find the work done by F along C.

$W = \int_C F \cdot dr$

$P_y = \cos xy = Q_x$ so the line integral is independent of path

let's find f s.t. $\nabla f = \langle P, Q \rangle$

since $P = f_x$, $f = \int P\,dx = \sin xy - x + g(y)$

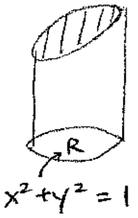
$f_y = x \cos xy + g'(y) = Q \Rightarrow g'(y) = 1, \; g(y) = y$

$\Rightarrow f = \sin xy - x + y$

$\int_C F \cdot dr = f(r(\pi/2)) - f(r(0)) = f(0,1) - f(1,0) = 1 - (-1) = 2$

Work= 2

7. Find the surface area of the portion of the plane $z = x + 2y + 10$ that lies within the cylinder $x^2 + y^2 = 1$.



$$SA = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA$$

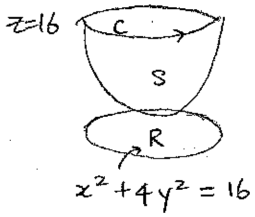
$$= \iint_R \sqrt{1+1+4}\ dA = \sqrt{6}\ \text{Area}(R) = \sqrt{6}\ \pi$$

OMG that was too easy

Surface area = $\sqrt{6}\ \pi$

8. Let C be the curve of intersection between the paraboloid $z = x^2 + 4y^2$ and the plane $z = 16$, traversed counterclockwise. Compute

$$\oint_C \underbrace{(3y + 4\cos x)}_{P}dx + \underbrace{(2x + 2\sin y)}_{Q}dy + \underbrace{(x^2/2 + 2y^2)}_{R}dz.$$



Since C is counterclockwise, S is oriented upward

by Stokes', $\oint_C F \cdot dr = \iint_S \text{curl} F \cdot n\, dS$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle 4y, -x, 2-3 \rangle$$

the surface S is given by $G(x,y,z) = z - x^2 - 4y^2$

so $n\,dS = \nabla G \, dA = \langle -2x, -8y, 1 \rangle \, dA$

$$\iint_S \text{curl} F \cdot n\, dS = \iint_R \langle 4y, -x, -1 \rangle \cdot \langle -2x, -8y, 1 \rangle \, dA$$

$$= \iint_R (-8xy + 8xy - 1)\, dA$$

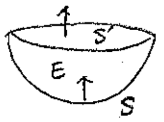
$$= -1 \text{ Area } (R)$$

R is the ellipse $x^2 + 4y^2 = 16$, or $\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1$

-1 Area (R) $= -8\pi$

Answer = -8π

9. Let $F = \langle z^3 \sin e^y, z^3 \cos e^x, y^2 + z\rangle$, and let S be the bottom half of the sphere $x^2 + y^2 + z^2 = 1$, oriented upward. Find the flux of F through S.



let E be the solid half ball bounded by S and S'.

since S is oriented upward, its unit normal actually points inward when considered as part of the closed surface S∪S'.

let S' be oriented upward

$$\text{flux} = \iint_S F\cdot n\, dS$$

by the Divergence Thm, $\displaystyle\iint_{\underset{\uparrow}{-S\cup S'}} F\cdot n\, dS = \iiint_E \text{div}\, F\, dV$

since the Divergence Thm applies to outward orientation, we have to flip the orientation of S

$$\text{so} \iiint_E \text{div}\, F\, dV = \iint_{S'} F\cdot n\, dS + \iint_{-S} F\cdot n\, dS$$

$$= \iint_{S'} F\cdot n\, dS - \iint_S F\cdot n\, dS$$

$$\iint_S F\cdot n\, dS = \iint_{S'} F\cdot n\, dS - \iiint_E \text{div}\, F\, dV$$

$$\iiint_E \text{div}\, F\, dV = \iiint_E 1\, dV = \text{vol}(E) = \frac{2\pi}{3}$$

$$\iint_{\underset{\substack{\uparrow \\ \text{circle of} \\ \text{radius 1} \\ \text{in the xy-plane}}}{S'}} F\cdot \underset{\substack{\uparrow \\ \langle 0,0,1\rangle}}{n}\, dS = \iint_{S'} (y^2 + \underset{\substack{\uparrow \\ 0}}{z})\, dA = \int_0^{2\pi}\int_0^1 r^2 \sin^2\theta\, r\, dr\, d\theta = \underbrace{\int_0^1 r^3\, dr}_{\left[\frac{r^4}{4}\right]_0^1 = \frac{1}{4}} \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} (1 - \cos(2\theta))\, d\theta$$

$$= \frac{1}{2}\left(2\pi - \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right) = \pi$$

$$\downarrow$$

$$= \frac{\pi}{4}$$

$$\text{hence} \iint_S F\cdot n\, dS = \frac{\pi}{4} - \frac{2\pi}{3} = \frac{3\pi}{12} - \frac{8\pi}{12} = \frac{-5\pi}{12}$$

Flux= $\frac{-5\pi}{12}$

10. (5 questions) Circle the best answer. **You do not need to show work.**

1. Two distinct solutions, X_1 and X_2, can be found to the system of equations $AX = B$. Which of the following is necessarily true?

   a. $B = \vec{0}$.

   b. A is invertible.

   c. A has more columns than rows.

   d. $X_1 = -X_2$.

   [✓ e.] There exists a solution X_3 that is different from both X_1 and X_2.

   $AX = B$ has either 0, 1, or infinitely many solutions

2. Let A, B, and C be 2-by-2 matrices, and let I denote the 2-by-2 identity matrix. Which of the following is/are always true?

   I. If A^2 is the zero matrix, then A is the zero matrix.

   II. If $AB = AC$, then $B = C$.

   III. If A is invertible and $A = A^{-1}$, then either $A = I$ or $A = -I$.

   a. III only.

   b. I and III only.

   c. II and III only.

   d. All of the above.

   [✓ e.] None of the above.

   counterexample for I : $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

   II is only necessarily true when A is invertible

   counterexample for III : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. Let A be an invertible matrix. If V is an eigenvector of A, which of the following is/are necessarily true?

   I. V is an eigenvector of A^{-1}.

   II. V is an eigenvector of A^2.

   III. V is an eigenvector of $2A$.

   a. I only.

   b. I and II only.

   c. I and III only.

   [✓ d.] All of the above.

   e. None of the above.

   if $AV = \lambda V$

   I. $A^{-1}AV = A^{-1}\lambda V \Rightarrow \lambda A^{-1}V = V \Rightarrow A^{-1}V = \frac{1}{\lambda} V$

   II. $A^2V = A\lambda V = \lambda AV = \lambda^2 V$

   III. $2AV = 2\lambda V \Rightarrow (2A)V = (2\lambda)V$

4. Let $f(x, y, z)$ and $g(x, y, z)$ be functions with continuous second partial derivatives. What is curl(grad$(f+g)$)?

a. 0.

[✓ b.] $\vec{0}$.

c. $f+g$.

d. (grad f)×(grad g).

e. Undefined.

$f+g$ is still a function with continuous second partials, for all such functions h, curl(grad h) = $\vec{0}$

5. Suppose that $P(x, y), Q(x, y), P_y$, and Q_x are continuous on $\mathbb{R}^2$ except at the origin, and that $P_y = Q_x$ (where defined). Suppose also that C_1 and C_2 are two piecewise smooth simple closed curves, both traversed counterclockwise. Which of the following is/are necessarily true?

I. If both C_1 and C_2 enclose the origin, then $\oint_{C_1} P dx + Q dy = \oint_{C_2} P dx + Q dy$.

II. If neither C_1 nor C_2 enclose the origin, then $\oint_{C_1} P dx + Q dy = \oint_{C_2} P dx + Q dy. = 0$

III. If C_1 encloses the origin and C_2 does not, then $\oint_{C_1} P dx + Q dy = -\oint_{C_2} P dx + Q dy$.

a. I only.

b. II only.

c. III only.

[✓ d.] I and II only.

e. All of the above.

Optional Question The following question has no specified point value, and you should not attempt it unless you have finished the rest of the exam. It will be used for extra credit considerations when determining your overall grade.

For an n-by-n matrix A, the *trace* of A is the sum of the entries on the main diagonal, i.e.

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

a. Prove that for any n-by-n matrix A, $\mathrm{tr}(-A)=-\mathrm{tr}(A)$.

b. Prove that for any n-by-n matrices A and B, $\mathrm{tr}(A+B)=\mathrm{tr}(A)+\mathrm{tr}(B)$.

c. Prove that for any n-by-n matrices A and B, $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

d. Show that there are no finite-dimensional square matrices A and B such that $AB - BA = I$.

(If A and B are infinite matrices, then $AB - BA = I$ can happen, which in quantum mechanics basically says that the position matrix A and the momentum matrix B do not commute in a serious way. From this one can derive Heisenberg's uncertainty principle.)

a. $\mathrm{tr}(-A) = \sum_{i=1}^{n} -A_{ii} = -\sum_{i=1}^{n} A_{ii} = -\mathrm{tr}(A)$

b. $\mathrm{tr}(A+B) = \sum_{i=1}^{n} (A_{ii}+B_{ii}) = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B)$

c. $\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ji}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji}A_{ij} = \mathrm{tr}(BA)$

d. if $AB - BA = I$, then $\mathrm{tr}(AB-BA) = \mathrm{tr}(I)$

$$\mathrm{tr}(AB-BA) \underset{\substack{\uparrow \\ \text{by (b)}}}{=} \mathrm{tr}(AB) + \mathrm{tr}(-BA) \underset{\substack{\uparrow \\ \text{by (a)}}}{=} \mathrm{tr}(AB) - \mathrm{tr}(BA) \underset{\substack{\uparrow \\ \text{by (c)}}}{=} \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0$$

but $\mathrm{tr}(I) = n$, so we cannot have $AB - BA = I$.